

PROPERTY A FOR COARSE SPACES

HIROKI SAKO

ABSTRACT. Property A introduced by Guoliang Yu is an amenability-type property for metric spaces. In this article, we study property A for uniformly locally finite coarse spaces. Main examples of coarse spaces are a metric space, a set equipped with a discrete group action, and a sequence of finitely generated groups. The purpose of this article is to give complete proofs to related basic facts.

1. INTRODUCTION

The subject of this article is amenability for generalized metric spaces. Property A for discrete metric spaces was introduced by G. Yu in his study of coarse Baum–Connes conjecture [Yu00]. Property A is widely recognized as an amenability-type condition. For discrete groups, amenability depends only on its large scale structure. Property A is also independent of local features of metric spaces, so it is natural to consider a new concept of spaces. J. Roe introduced a notion called coarse space in [Roe03]. The category of coarse spaces encompasses

- a metric space;
- a discrete group;
- a set equipped with a discrete group action (Example 2.3);
- a sequence of finite generated groups with fixed generators (Example 2.4).

In this article, we clarify the definition of property A for uniformly locally finite coarse spaces, which is not necessarily metrizable. Uniform local finiteness of a coarse space X means that X has very poor local structure. We prove that the property can be rephrased in the following context:

- a characterization by Hilbert space;
- nuclearity of the uniform Roe algebra;
- the operator norm localization property (ONL).

This article is not a usual research paper. Ideas in the proofs have already appeared in papers which dealt with property A for metric spaces ([BNW07], [CTWY08], [HR00], [STY02], [Roe03], [Sak12a]). We give a precise proof for the equivalence between property A and ONL, which has been proved only for metric spaces. The general form (Theorem 5.7) of the theorem plays a key role in the paper [Sak12b], because the argument is not closed in the category of metric spaces. The framework of coarse spaces is not only necessary but also appropriate.

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2. DEFINITION OF PROPERTY A

2.1. Coarse space. A coarse space is a set X equipped with a coarse structure \mathcal{C} . The coarse structure \mathcal{C} is a family of subsets of X^2 and satisfies several requirements. To describe them, we will use the following notations. For subsets $T, T_1, T_2 \subset X^2$, the inverse T^{-1} and the product $T_1 \circ T_2$ are defined as follows:

$$\begin{aligned} T^{-1} &= \{(x, y) \in X^2 ; (y, x) \in T\}, \\ T_1 \circ T_2 &= \{(x, y) \in X^2 ; \exists z \in X, (x, z) \in T_1, (z, y) \in T_2\}. \end{aligned}$$

Denote by $T^{\circ n}$ the n -th power $T \circ T \circ \cdots \circ T$. In the case that the subsets are graphs of partially defined maps, these notations coincide with the usual conventions for mappings. For subsets $Y \subset X$ and $T \subset X^2$, let $T[Y]$ be the subset of X defined by

$$\{x \in X ; \exists y \in Y, (x, y) \in T\}.$$

For a singleton $\{x\}$, we simply write $T[x]$ for $T[\{x\}]$. A subset $Y \subset X$ is called a T -bounded set if there exists $x \in X$ such that $Y \subset T[x]$. If Y is a T_2 -bounded set, then $T_1[Y]$ is a $(T_1 \circ T_2)$ -bounded set.

Definition 2.1 (Definition 2.3 in [Roe03]). *Let X be a set. A family \mathcal{C} of subsets of X^2 is called a coarse structure on X if*

- $\Delta_X = \{(x, x); x \in X\} \in \mathcal{C}$;
- If $T \in \mathcal{C}$, then $T^{-1} \in \mathcal{C}$;
- If $T_1, T_2 \in \mathcal{C}$, then $T_1 \circ T_2 \in \mathcal{C}$;
- If $T_1, T_2 \in \mathcal{C}$, then $T_1 \cup T_2 \in \mathcal{C}$;
- If $T_1 \in \mathcal{C}$ and $T_2 \subset T_1$, then $T_2 \in \mathcal{C}$.

The pair (X, \mathcal{C}) is called a coarse space. Elements of \mathcal{C} are called controlled sets or entourages.

Two elements $x, y \in X$ are said to be connected if $\{(x, y)\} \in \mathcal{C}$. If arbitrary two points are connected, the space X is said to be connected.

Example 2.2. Let (X, d) be a metric spaces. A coarse structure \mathcal{C} is defined on X by $\mathcal{C} = \{T \subset X^2 ; \exists S > 0, d|_T \leq S\}$. The space X is connected.

Example 2.3. Let X be a set on which a discrete group G acts. For a finite subset $K \subset G$, let T_K be the orbit of K . That is, $T_K = \{(kx, x) \in X^2; k \in K, x \in X\}$. The coarse structure \mathcal{C}_G on X is the collection $\{T; \exists \text{ finite } K \subset G, T \subset T_K\}$. If G is countable and the action on X is transitive, then the coarse structure is realized by some metric. The group G naturally has a coarse structure defined by the left transformation action of G .

Example 2.4. Let $\left\{ \left(G^{(m)}, g_1^{(m)}, g_2^{(m)}, \dots, g_k^{(m)} \right) \right\}_{m=1}^{\infty}$ be a sequence of groups with fixed k -generators. Define surjective homomorphisms $\phi^{(m)}: F_k \rightarrow G^{(m)}$ from the free group $F_k = \langle g_1, \dots, g_k \rangle$ by $\phi^{(m)}(g_i) = g_i^{(m)}$. The set $\bigsqcup_{m=1}^{\infty} G^{(m)}$ is equipped with an F_k -action defined by $h \cdot g = \phi^{(m)}(h)g, g \in G^{(m)}$. This action gives a coarse structure on the disjoint union of groups $\bigsqcup_{m=1}^{\infty} G^{(m)}$. This coarse space is not connected.

Definition 2.5. A coarse space (X, \mathcal{C}) is said to be uniformly locally finite if every controlled set $T \in \mathcal{C}$ satisfies the inequality $\sup_{x \in X} \sharp(T[x]) < \infty$.

The coarse spaces in Example 2.3 and Example 2.4 are uniformly locally finite. In many references, a metric space whose coarse structure is uniformly locally finite is called a metric space with bounded geometry.

2.2. Definition of property A. The definition of property A for X is given by a Følner condition of $X \times \mathbb{N}$.

Definition 2.6 (Definition 2.1 of Yu [Yu00]). A discrete metric space X is said to have property A if for every $\epsilon > 0$ and $R > 0$, there exist $S > 0$ and a family of finite subsets $\{A_x\}_{x \in X}$ of $X \times \mathbb{N}$ such that

- $A_x \subset \{y \in X; d(x, y) < S\} \times \mathbb{N}$;
- $(x, 1) \in A_x$;
- The symmetric difference $A_x \triangle A_y = (A_x \setminus A_y) \cup (A_y \setminus A_x)$ satisfies the inequality $\sharp(A_x \triangle A_y) < \epsilon \sharp(A_x \cap A_y)$, when $d(x, y) \leq R$.

We define property A for coarse spaces as follows.

Definition 2.7. A uniformly locally finite coarse space (X, \mathcal{C}) is said to have property A if for every positive number ϵ and every controlled set $T \in \mathcal{C}$, there exist a controlled set $S \in \mathcal{C}$ and a subset $A \subset S \times \mathbb{N}$ such that

- For $x \in X$, $A_x = \{(y, n) \in X \times \mathbb{N}; (x, y, n) \in A\}$ is finite;
- $\Delta_X \times \{1\} \subset A$, where Δ_X is the diagonal subset of X^2 ;
- $\sharp(A_x \triangle A_y) < \epsilon \sharp(A_x \cap A_y)$, if $(x, y) \in T$.

The second condition can be replaced with the following:

- For $x \in X$, A_x is not empty.

This is a conclusion of the proof of Theorem 3.1.

3. CHARACTERIZATIONS BY HILBERT SPACES AND POSITIVE DEFINITE KERNELS

Let X be a uniformly locally finite coarse space. For a controlled set $T \subset X^2$ of X , we define a linear space E_T of bounded linear operators on $\ell^2(X)$ as follows:

$$E_T = \{a \in \mathbb{B}(\ell^2(X)); \langle a\delta_y, \delta_x \rangle = 0 \text{ if } (x, y) \in X^2 \setminus T\}.$$

We note that

- if $b_1 \in E_{T_1}$ and $b_2 \in E_{T_2}$, then $b_1 b_2 \in E_{T_1 \circ T_2}$;
- and that if $b \in E_T$, then $b^* \in E_{T^{-1}}$.

We also define a C*-algebra called the uniform Roe algebra $C_u^*(X)$ of X , which plays a key role for characterizations of property A. The algebra $C_u^*(X)$ is defined by $\overline{\bigcup_{T: \text{controlled}} E_T}$.

Theorem 3.1. For a uniformly locally finite coarse space (X, \mathcal{C}) , the following conditions are equivalent:

- (1) X has property A;

- (2) For every $\epsilon > 0$, and every $T \in \mathcal{C}$, there exists a map $\eta: X \rightarrow \ell^2(X)$ assigning unit vectors such that
- if $(x, y) \in T$, then $\|\eta_x - \eta_y\| < \epsilon$,
 - $\{(x, y) \in X^2; y \in \text{supp}(\eta_x)\}$ is controlled;
- (3) For every $\epsilon > 0$, and every $T \in \mathcal{C}$, there exist a Hilbert space \mathcal{H} and a map $\eta: X \rightarrow \mathcal{H}$ assigning unit vectors such that
- if $(x, y) \in T$, then $\|\eta_x - \eta_y\| < \epsilon$,
 - $\{(x, y) \in X^2; \langle \eta_y, \eta_x \rangle \neq 0\}$ is controlled;
- (4) For every $\epsilon > 0$ and every $T \in \mathcal{C}$, there exists a positive definite kernel $k: X^2 \rightarrow \mathbb{C}$ such that
- if $(x, y) \in T$, then $\|1 - k(x, y)\| < \epsilon$,
 - $\{(x, y) \in X^2; k(x, y) = 0\}$ is controlled.

For basic facts of positive definite kernel (or function of positive type), see Appendix C of the book [BdlHV08]. This theorem for metric spaces is given in [BNW07, Theorem 3].

Proof. Suppose that X has property A. Let ϵ be an arbitrary positive number and let T be a controlled set of X . Then there exist a controlled set S and a subset $A \subset S \times \mathbb{N}$ which satisfy the conditions of Definition 2.7. Define a subset $A_x(y)$ of \mathbb{N} by

$$A_x(y) = \{n \in \mathbb{N}; (y, n) \in A\} = \{n \in \mathbb{N}; (x, y, n) \in A\}.$$

Let $\|\cdot\|_1$ denote the norm of $\ell^1(X)$. For $x \in X$, define vectors ζ_x, ξ_x of $\ell^1(X)$ by

$$\zeta_x(y) = \sharp(A_x(y)), \quad \xi_x = \zeta_x / \|\zeta_x\|_1.$$

For $(x, y) \in T$, we have

$$\begin{aligned} \|\xi_x - \xi_y\|_1 &= \frac{\|\|\zeta_y\|_1 \zeta_x - \|\zeta_x\|_1 \zeta_y\|_1}{\|\zeta_x\|_1 \|\zeta_y\|_1} \\ &\leq \frac{\|\|\zeta_y\|_1 - \|\zeta_x\|_1\| \cdot \|\zeta_x\|_1 + \|\zeta_x\|_1 \cdot \|\zeta_x - \zeta_y\|_1}{\|\zeta_x\|_1 \|\zeta_y\|_1} \\ &\leq 2 \frac{\|\zeta_x - \zeta_y\|_1}{\|\zeta_y\|_1} \\ &\leq 2 \frac{\sharp(A_x \triangle A_y)}{\sharp(A_y)} \\ &< 2\epsilon. \end{aligned}$$

Define a unit vector η_x of $\ell^2(X)$ by $\eta_x(y) = \sqrt{\xi_x(y)}$. Then we have

$$\|\eta_x - \eta_y\|_2 \leq \sqrt{\|\xi_x - \xi_y\|_1} < \sqrt{2\epsilon}, \quad (x, y) \in T.$$

The set $\{(x, y) \in X^2; y \in \text{supp}(\eta_x)\}$ is controlled, since it is included in S . Here we obtain condition (2).

When we denote by S the subset $\{(x, y) \in X^2; y \in \text{supp}(\eta_x)\}$, the subset $\{(x, y) \in X^2; \langle \eta_y, \eta_x \rangle \neq 0\}$ is included in $S \circ S^{-1}$. Therefore (2) implies (3).

Suppose that (3) holds. For arbitrary $\epsilon > 0$ and $T \in \mathcal{C}$, there exists a map $\eta: X \rightarrow \mathcal{H}$ such that $\|\eta_x\| = 1$ and

- if $(x, y) \in T$, then $\|\eta_x - \eta_y\| < \epsilon$,
- $\{(x, y) \in X^2; \langle \eta_x, \eta_y \rangle \neq 0\}$ is controlled.

Define a function $k: X^2 \rightarrow \mathbb{C}$ by $k(x, y) = \langle \eta_x, \eta_y \rangle$. The function k satisfies condition (4).

For the converse direction, we first prove that condition (4) implies condition (2). Suppose that condition (4) holds. For arbitrary $\epsilon > 0$ and arbitrary controlled set T , take a positive definite kernel $k: X^2 \rightarrow \mathbb{C}$ satisfying (4). Then there exists an operator a in $C_u^*(X)$ such that $\langle a\delta_y, \delta_x \rangle = k(x, y)$. Note that the operator a is positive. So there exist $S \in \mathcal{C}$ and $b \in E_S$ which satisfies $\|a - b^*b\| < \epsilon$. For $x \in X$, we define a vector $\zeta_x \in \ell^2(X)$ by $b\delta_x$. Since $\{(x, y) \in X^2; y \in \text{supp}(\zeta_x)\}$ is included in S^{-1} , it is controlled. For $(x, y) \in T$, we have

$$\begin{aligned} \|\zeta_x - \zeta_y\|^2 &= k(x, x) - 2\text{Re}(k(x, y)) + k(y, y) < 4\epsilon, \\ |1 - \|\zeta_x\|^2| &= |1 - k(x, x)| < \epsilon. \end{aligned}$$

When we define η_x by $\zeta_x / \|\zeta_x\|$, $\|\eta_x - \eta_y\|$ is uniformly close to 0 for $(x, y) \in T$. Thus we obtain condition (2).

Finally, we prove that condition (2) implies condition (1). For arbitrary $\epsilon > 0$ and $T \in \mathcal{C}$, take unit vectors $\{\eta_x\}_{x \in X}$ satisfying condition (2). We define a positive element ξ_x of $\ell^1(X)$ by $\xi_x(y) = |\eta_x(y)|^2$. For $(x, y) \in T$, we have $\|\xi_x - \xi_y\|_1 \leq \|\eta_x + \eta_y\|_2 \|\eta_x - \eta_y\|_2 < 2\epsilon$. The cardinality of the support $\text{supp}(\xi_x)$ is uniformly bounded. More precisely, $\sup_{x \in X} \#(\text{supp}(\xi_x)) < \infty$. By replacing ξ_x , we obtain the following: there exists a natural number m such that

- $|1 - \|\xi_x\|_1| < \epsilon$;
- If $(x, y) \in T$, then $\|\xi_x - \xi_y\|_1 < \epsilon$;
- $\{(x, y) \in X^2; y \in \text{supp}(\xi_x)\}$ is controlled;
- For every $(x, y) \in X^2$, $\xi_x(y) \in \{n/m; n \in \mathbb{N} \cup \{0\}\}$;
- $\xi_x(x) \neq 0$.

Let S denote the controlled set $\{(x, y) \in X^2; y \in \text{supp}(\xi_x)\}$. Define $A \subset S \times (\mathbb{N} \cup \{0\})$ by

$$A = \{(x, y, n); n < m \cdot \xi_x(y)\}.$$

Then the family of the finite sets $\{A_x = \{(y, n); (x, y, n) \in A\}\}_{x \in X}$ satisfy

$$\#(A_x \triangle A_y) = \|m\xi_x - m\xi_y\|_1 < \epsilon m < \frac{\epsilon}{1 - \epsilon} \#(A_x), \quad (x, y) \in T.$$

We also have

- $\#(A_x \triangle A_y) < 2\epsilon \#(A_x \cap A_y) / (1 - \epsilon)$, $(x, y) \in T$;
- $(x, 0) \in A_x$.

It follows that X has property A. □

4. CHARACTERIZATION BY TRANSLATION C^* -ALGEBRAS

We generalize the easier half of Theorem 5.3 in [STY02] in this section.

Lemma 4.1. *Let X be a set and let T be a subset of X^2 . Suppose that $\sharp(T[x]), \sharp(T^{-1}[x])$ are uniformly bounded. Let b be an operator whose matrix coefficients are zero on $X^2 \setminus T$. Then we have the following estimate of the operator norm:*

$$\|b\| \leq \max \left\{ \sup_{x \in X} \sharp(T[x]), \sup_{x \in X} \sharp(T^{-1}[x]) \right\} \cdot \sup_{x, y \in X} |\langle b\delta_y, \delta_x \rangle|.$$

This lemma is one version of the Schur tests.

Proof. We denote by $b_{x,y}$ the matrix coefficient $\langle b\delta_y, \delta_x \rangle$. Let $\xi = (\xi_x)_{x \in X}$ and $\eta = (\eta_x)_{x \in X}$ be elements of $\ell^2(X)$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\langle b\eta, \xi \rangle| &= \left| \sum_{x,y} \bar{\xi}_x b_{x,y} \eta_y \right| \\ &\leq \sum_{x,y} |\xi_x| |b_{x,y}|^{1/2} |b_{x,y}|^{1/2} |\eta_y| \\ &\leq \left(\sum_{x,y} |\xi_x|^2 |b_{x,y}| \right)^{1/2} \left(\sum_{x,y} |b_{x,y}| |\eta_y|^2 \right)^{1/2} \\ &\leq \sup_{x,y} |b_{x,y}| \left(\sup_x \sharp(T^{-1}[x]) \sum_x |\xi_x|^2 \right)^{1/2} \left(\sup_y \sharp(T[y]) \sum_y |\eta_y|^2 \right)^{1/2} \\ &\leq \max \left\{ \sup_{y \in X} \sharp(T[y]), \sup_{x \in X} \sharp(T^{-1}[x]) \right\} \cdot \sup_{x,y} |b_{x,y}| \cdot \|\xi\| \cdot \|\eta\|. \end{aligned}$$

Thus we obtain the inequality in the lemma. \square

Theorem 4.2. *Let X be a uniformly locally finite coarse space. If X has property A, then the translation C^* -algebra $C_u^*(X)$ is nuclear.*

The proof is given by a verbatim translation of [Roe03, Proposition 11.41]. For a controlled set $S \subset X^2$, we denote by C_S the C^* -algebra $\prod_{z \in X} \mathbb{B}(\ell^2(S[z]))$. We also define a unital completely positive map $\Phi_S : C_u^*(X) \rightarrow C_S$ by

$$\Phi_S(b) = [[b_{x,y}]_{x,y \in S[z]}]_{z \in X},$$

where $b_{x,y}$ is the matrix coefficient $\langle b\delta_y, \delta_x \rangle$.

Proof. For any positive number ϵ and any finite subset \mathcal{F} of $C_u^*(X)$, we find a controlled set S and construct a unital completely positive map $\Psi : C_S \rightarrow C_u^*(X)$ such that $\|\Psi \circ \Phi_S(b) - b\| < \epsilon$, for $b \in \mathcal{F}$. Since C_S is nuclear, the construction provides the proof. By approximating elements of \mathcal{F} , we may assume that for every $b \in \mathcal{F}$ the supports of the matrix coefficients $[b_{x,y} = \langle b\delta_y, \delta_x \rangle]_{x,y \in X}$ are included in a controlled set T .

Define $\delta > 0$ by $\frac{\epsilon}{\max_{b \in \mathcal{F}} \|b\| \cdot \max\{\sup \sharp(T[x]), \sup \sharp(T^{-1}[x])\}}$. By Theorem 3.1, there exists a map $\eta : X \rightarrow \ell^2(X)$ assigning unit vectors such that

- If $(x, y) \in T$, then $\|\eta_x - \eta_y\| < \delta$;
- $S = \{(x, z) \in X^2; z \in \text{supp}(\eta_x)\}$ is controlled.

Define $\Psi : C_S \rightarrow C_u^*(X)$ by

$$\Psi([c_{x,y}^{(z)}]_{x,y \in S[z]}]_{z \in X}) = \left[\sum_{z \in X} \overline{\eta_x(z)} c_{x,y}^{(z)} \eta_y(z) \right]_{x,y \in X}.$$

If $x \notin S[z]$ or if $y \notin S[z]$, we define $c_{x,y}^{(z)}$ by 0. It is routine to prove that Ψ is unital and completely positive. For $b \in \mathcal{F}$, the operator $\Psi \circ \Phi_S(b)$ is expressed as

$$\left[\sum_{z \in X} \overline{\eta_x(z)} b_{x,y} \eta_y(z) \right]_{x,y \in X} = [\langle \eta_y, \eta_x \rangle b_{x,y}]_{x,y \in X}.$$

Therefore the matrix coefficients of $\Psi \circ \Phi_S(b)$ are not zero only on the controlled set T . For $(x, y) \in T$, $|1 - \langle \eta_y, \eta_x \rangle| = |\langle \eta_y, \eta_y - \eta_x \rangle| < \delta$. It follows that

$$\begin{aligned} \|\Psi \circ \Phi_S(b) - b\| &= \|(1 - \langle \eta_y, \eta_x \rangle) b_{x,y}\|_{x,y \in X} \\ &\leq \max\{\sup \#(T[x]), \sup \#(T^{-1}[x])\} \cdot \sup_{(x,y) \in T} |(1 - \langle \eta_y, \eta_x \rangle) b_{x,y}| \\ &< \max\{\sup \#(T[x]), \sup \#(T^{-1}[x])\} \cdot \delta \|b\| \\ &< \epsilon. \end{aligned}$$

The second line is due to Lemma 4.1. \square

The converse is also true. In the paper [Sak12b], the author proves the following stronger claim:

Theorem 4.3. *Local reflexivity of $C_u^*(X)$ is equivalent to property A of X .*

Local reflexivity is weaker than exactness and nuclearity for general C^* -algebras. As a consequence, all the following properties of $C_u^*(X)$ are equivalent: nuclearity, exactness, and local reflexivity.

5. CHARACTERIZATION BY THE OPERATOR NORM LOCALIZATION

Roughly speaking, Theorem 3.1 means that a space X has property A if and only if its large scale structure can be described by a Hilbert space. The coarse amenability of X can be further rephrased in terms of operator norms.

5.1. Definitions of the operator norm localization property. Chen, Tessera, Wang and Yu defined the operator norm localization property in [CTWY08, section 2]. The original definition is given for metric spaces. For a general uniformly locally finite coarse space X , we define the property as follows.

Definition 5.1. *A uniformly locally finite coarse space (X, \mathcal{C}) is said to have the operator norm localization property (ONL) if for every $c < 1$ and $T \in \mathcal{C}$, there exists a controlled set S satisfying condition (β) : for every operator $a \in E_T$, there exists a unit vector $\eta \in \ell^2(X)$ such that $\text{supp}(\eta)$ is an S -bounded set and $c\|a\| \leq \|\eta a\|$.*

This definition is analogous to condition (iv) in [Sak12a, Proposition 3.1]. We may replace ‘for every $c < 1$ ’ with ‘there exists $c < 1$.’

Lemma 5.2. *A uniformly locally finite coarse space X has ONL, if and only if there exists $0 < c < 1$ such that for every controlled set T , there exists a controlled set S satisfying condition (β) .*

Following an idea in [CTWY08, Proposition 2.4], we give a proof.

Proof. Assume that X satisfies the property in Lemma 5.2 with respect to a constant $c < 1$. Let κ be an arbitrary real number less than 1 and let T be an arbitrary controlled set. Replacing T by $\Delta_X \cup T \cup T^{-1}$, we may assume that T is symmetric and includes Δ_X . Choose a natural number n satisfying $\kappa^n < c$.

By the assumption on X , there exists S satisfying the following condition: for every $b \in E_{T \circ 2n}$, there exists a unit vector $\xi \in \ell^2(X)$ such that $\text{supp}(\xi)$ is an S -bounded set and $c\|b\| \leq \|b\xi\|$. Let $a \in E_T$ be an arbitrary operator of norm 1. Since the matrix coefficients of $(aa^*)^n$ are located on $T^{\circ 2n}$, there exists a unit vector $\xi \in \ell^2(X)$ such that $\text{supp}(\xi)$ is S -bounded and that $c\|(aa^*)^n\| \leq \|(aa^*)^n\xi\|$. Since the norm of $(aa^*)^n$ is 1, we have

$$\kappa^n < c \leq \frac{\|(aa^*)^n\xi\|}{\|(aa^*)^{n-1}\xi\|} \cdots \frac{\|(aa^*)^2\xi\|}{\|(aa^*)\xi\|} \frac{\|(aa^*)\xi\|}{\|\xi\|}.$$

It follows that there exists $j = 0, 1, \dots, n-1$ such that $\kappa < \|(aa^*)^{j+1}\xi\|/\|(aa^*)^j\xi\|$. By letting $\eta = a^*(aa^*)^j\xi/\|a^*(aa^*)^j\xi\|$, we have the inequality

$$\kappa\|a\| = \kappa < \|(aa^*)^{j+1}\xi\|/\|(aa^*)^j\xi\| \leq \|aa^*(aa^*)^j\xi\|/\|a^*(aa^*)^j\xi\| = \|a\eta\|.$$

The support of $a^*(aa^*)^j\xi$ is $(T^{\circ 2n-1} \circ S)$ -bounded. Thus we obtain condition (β) for κ , T , and $T^{\circ 2n-1} \circ S$. \square

To characterize ONL, we use ‘ E_T ’ and ‘ Φ_S ’ in the proof of Theorem 4.2.

Lemma 5.3. *A uniformly finite coarse space X has ONL if and only if the following condition holds: For every $\epsilon > 0$ and a controlled set T , there exists a controlled set S satisfying $\|(\Phi_S|_{E_T})^{-1}: \Phi_S(E_T) \rightarrow E_T\| < 1 + \epsilon$.*

Proof. Assume that X has ONL. For arbitrary $\epsilon > 0$ and a controlled set T , there exists S which satisfies the condition (β) for $c = (1 + \epsilon)^{-1}$ and T .

It follows that for every non-zero operator $a \in E_T$, there exists a unit vector $\eta \in \ell^2(X)$ whose support is S -bounded and satisfies $\|a\| \leq (1 + \epsilon)\|a\eta\|$. Since $a \in E_T$, $\text{supp}(a\eta)$ is included in the subset $T[\text{supp}(\eta)]$. Hence there exists a unit vector ξ such that $\|a\eta\| = \langle a\eta, \xi \rangle$ and that the supports of ξ , η are included in a common $(T \circ S)$ -bounded set. There exists $x \in X$ satisfying

$$\|a\| \leq (1 + \epsilon)\langle a\eta, \xi \rangle \leq (1 + \epsilon)\|[a_{y,z}]_{y,z \in T \circ S[x]}\| \leq (1 + \epsilon)\|\Phi_{T \circ S}(a)\|,$$

we get $\|(\Phi_{T \circ S}|_{E_T})^{-1}\| \leq 1 + \epsilon$.

Conversely, suppose that for every $c < 1$ and a controlled set T , there exists a controlled set S satisfying $\|(\Phi_S|_{E_T})^{-1}: \Phi_S(E_T) \rightarrow E_T\| < 1/c$. Then for every operator $a \in E_T$, there exists an S -bounded set $S[x]$ satisfying

$$c\|a\| \leq \|[a_{y,z}]_{y,z \in S[x]}\|.$$

Take a unit vector $\eta \in \ell^2(S[x])$ such that $\|[a_{y,z}]_{y,z \in S[x]}\| = \|[a_{y,z}]_{y,z \in S[x]}\eta\|$. The vector η satisfies $c\|a\| \leq \|a\eta\|$. It follows that condition (β) holds true for $c < 1$, T , and S . We conclude that X has ONL. \square

Let us recall the notions of completely positive map and completely bounded map.

Definition 5.4. *A closed subspace E of a unital C^* -algebra is called an operator system, if*

- *For every element a of E , a^* is also an element of E ;*
- *The unit of the ambient C^* -algebra is an element of E .*

A linear map Φ from E to a C^ -algebra C is said to be completely positive, if the map $\Phi^{(n)} = \Phi \otimes \text{id}: E \otimes \mathbb{M}_n(\mathbb{C}) \rightarrow C \otimes \mathbb{M}_n(\mathbb{C})$ is positive for every n .*

For controlled set $T \subset S$ The subspaces $E_T \subseteq C_u^*(X)$ and $\Phi_S(E_T) \subseteq D_S$ are examples of operator systems. The map $\Phi_S: E_T \rightarrow C_S$ is completely positive.

Definition 5.5. *Let B be a C^* -algebra and let F be an operator system. A linear map $\Psi: F \rightarrow B$ is said to be completely bounded if the increasing sequence $\{\|\Psi^{(n)}: F \otimes \mathbb{M}_n(\mathbb{C}) \rightarrow B \otimes \mathbb{M}_n(\mathbb{C})\|\}$ is bounded. The number $\|\Psi\|_{\text{cb}} = \sup_{n \in \mathbb{N}} \|\Psi^{(n)}\|$ is called the completely bounded norm of θ .*

The norms $\|\Psi\|$ and $\|\Psi\|_{\text{CB}}$ are not identical in general, but we have the following.

Lemma 5.6. $\|(\Phi_S|_{E_T})^{-1}: \Phi_S(E_T) \rightarrow E_T\|_{\text{CB}} = \|(\Phi_S|_{E_T})^{-1}\|$.

Proof. For a natural number n , we denote by $((\Phi_S|_{E_T})^{-1})^{(n)}$ the linear map

$$(\Phi_S|_{E_T})^{-1} \otimes \text{id}: \Phi_S(E_T) \otimes \mathbb{M}(n, \mathbb{C}) \rightarrow E_T \otimes \mathbb{M}(n, \mathbb{C}).$$

It suffices to show that

$$\|((\Phi_S|_{E_T})^{-1})^{(n)}\| \leq \|(\Phi_S|_{E_T})^{-1}\| \quad \text{for every } n \in \mathbb{N}.$$

Take an arbitrary positive number K satisfying $K < \|((\Phi_S|_{E_T})^{-1})^{(n)}\|$. There exists an operator $a \in E_T \otimes \mathbb{M}(n, \mathbb{C})$ satisfying

$$K < \|a\|, \quad \|\Phi_S \otimes \text{id}_{\mathbb{M}(n, \mathbb{C})}(a)\| = 1.$$

We claim that there exist isometries $V, W: \ell^2(X) \rightarrow \ell^2(X) \otimes \mathbb{C}^n$ satisfying

$$V\delta_x, W\delta_x \in \mathbb{C}\delta_x \otimes \mathbb{C}^n \quad \text{and} \quad K < \|W^*aV\| \leq \|a\|.$$

Indeed, there exist unit vectors $\xi = \sum \delta_x \otimes \xi_x$ and $\eta = \sum \delta_y \otimes \eta_x$ such that

$$K < |\langle a\xi, \eta \rangle| < \|a\|.$$

We define isometries V, W by

$$V(\delta_x) = \begin{cases} \delta_x \otimes \xi_x / \|\xi_x\|, & \text{if } \xi_x \neq 0, \\ \delta_x \otimes \delta_1, & \text{if } \xi_x = 0, \end{cases} \quad W(\delta_x) = \begin{cases} \delta_x \otimes \eta_x / \|\eta_x\|, & \text{if } \eta_x \neq 0, \\ \delta_x \otimes \delta_1, & \text{if } \eta_x = 0. \end{cases}$$

Then we have $K < |\langle a\xi, \eta \rangle| \leq \|W^*aV\| \leq \|a\|$.

Observe that the matrix coefficients of W^*aV are zero on $X^2 \setminus T$ and that

$$\|\Phi_S(W^*aV)\| \leq \|\Phi_S^{(n)}(a)\| = 1.$$

It follows that $K < \|W^*aV\|/\|\Phi_S(W^*aV)\| \leq \|(\Phi_S|_{E_T})^{-1}\|$. We obtain the inequality $\|((\Phi_S|_{E_T})^{-1})^{(n)}\| \leq \|(\Phi_S|_{E_T})^{-1}\|$. \square

5.2. Property A and ONL. The author proved in [Sak12a, Theorem 4.1] that the operator norm localization property is equivalent to property A for a metric space with bounded geometry. More generally, this theorem is also valid for uniformly locally finite coarse spaces.

Theorem 5.7. *A uniformly locally finite coarse space X has property A, if and only if X has the operator norm localization property.*

Proof. We first assume that X has property A. Take an arbitrary controlled set $T \subset X^2$ and $\epsilon > 0$. Define $\delta > 0$ by $\frac{\epsilon}{\max\{\sup \sharp(T[x]), \sup \sharp(T^{-1}[x])\}}$. By Theorem 3.1, there exist unit vectors $\{\eta_x\}_{x \in X} \subseteq \ell^2(X)$ satisfying the following:

- if $(x, y) \in T$, then $\|\eta_x - \eta_y\| < \epsilon$,
- $S = \{(x, z) \in X^2; z \in \text{supp}(\eta_x)\}$ is controlled.

Recall that the C*-algebra C_S is defined by $\prod_{z \in X} \mathbb{B}(\ell^2(S[z]))$. We also use the unital completely positive maps $\Phi_S : C_u^*(X) \rightarrow C_S$ and $\Psi : C_S \rightarrow C_u^*(X)$ in the proof of Theorem 4.2. For an operator $b \in E_T$, and the matrix coefficient of $\Psi \circ \Phi(b)$ at $(x, y) \in T$ is $\langle \eta_y, \eta_x \rangle b_{x,y}$. By Lemma 4.1, the following inequality follows:

$$\begin{aligned} \|\Psi \circ \Phi_S(b) - b\| &\leq \max\{\sup \sharp(T[x]), \sup \sharp(T^{-1}[x])\} \cdot \sup_{(x,y) \in T} |\langle \eta_y, \eta_x \rangle b_{x,y} - b_{x,y}| \\ &\leq \epsilon \|b\|. \end{aligned}$$

Since Ψ is contractive, we have $\|\Phi_S(b)\| \geq \|\Psi \circ \Phi_S(b)\| \geq (1 - \epsilon)\|b\|$. Then there exists $x \in X$ such that $\|b|_{\ell^2(S[x])}\| \geq (1 - 2\epsilon)\|b\|$. It follows that there exists a unit vector $\xi \in \ell^2(S[x])$ such that $\|b\xi\| \geq (1 - 2\epsilon)\|b\|$. We conclude that X has ONL.

Now assume that X has ONL. By Lemma 5.3 and Lemma 5.6, for any controlled set T and $\epsilon > 0$, there exists a controlled set S such that

$$\|(\Phi_S|_{E_T})^{-1} : \Phi_S(E_T) \rightarrow E_T\|_{\text{CB}} < 1 + \epsilon/2.$$

It is easy to check that $(\Phi_S|_{E_T})^{-1}$ is unital and self-adjoint. By Corollary B.9 of the book [BO08], there exists a unital completely positive map $\Psi : B_S \rightarrow \mathbb{B}(\ell^2(X))$ which satisfies $\|(\Phi_S|_{E_R})^{-1} - \Psi|_{\Phi_S(E_R)}\|_{\text{CB}} < \epsilon$.

Define a function k on the set X^2 by $k(x, y) = \langle \Psi \circ \Phi_S(e_{x,y})\delta_y, \delta_x \rangle$, where $e_{x,y}$ is the rank 1 partial isometry which maps δ_y to δ_x . Since $\Psi \circ \Phi_S$ is completely positive, for every $x(1), x(2), \dots, x(n) \in X$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$, we have

$$\begin{aligned} 0 &\leq \left\langle (\Psi \circ \Phi_S)^{(n)} \left(\begin{bmatrix} e_{x(1),x(1)} & \cdots & e_{x(1),x(n)} \\ \vdots & \ddots & \vdots \\ e_{x(n),x(1)} & \cdots & e_{x(n),x(n)} \end{bmatrix} \right) \begin{bmatrix} \lambda_1 \delta_{x(1)} \\ \vdots \\ \lambda_n \delta_{x(n)} \end{bmatrix}, \begin{bmatrix} \lambda_1 \delta_{x(1)} \\ \vdots \\ \lambda_n \delta_{x(n)} \end{bmatrix} \right\rangle \\ &= \sum_{i,j=1}^n \bar{\lambda}_i \lambda_j k(x(i), x(j)). \end{aligned}$$

It follows that k is a positive definite kernel on X . The support of k is included in the controlled set S , because $\Phi_S(e_{x,y}) = 0$ if $(x, y) \notin S$. For $(x, y) \in T$, we have

$$|1 - k(x, y)| = |\langle (e_{x,y} - \Psi \circ \Phi_S(e_{x,y}))\delta_y, \delta_x \rangle| \leq \|((\Phi_S|_{E_R})^{-1} - \Psi)(\Phi_S(e_{x,y}))\| < \epsilon.$$

It follows that X satisfies condition (4) in Theorem 3.1. \square

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: hiroki.sako@gmail.com